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Diffeomorphisms and orthonormal frames

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Abstract

There is a natural homomorphism of Lie pseudoalgebras from local vector fields to local rotations on a Riemannian manifold. We address the question whether this homomorphism is unique and give a positive answer in the perturbative regime around the flat metric.

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1. Introduction

The spin lift $SO(3) \to SU(2)$ is an important part of quantum mechanics. Indeed experiment teaches us that spinors, e.g. neutrons, undergoing a rotation by 360° differ from the non-rotated ones [8], and the action of the rotation group on spinors is double valued. In special relativity this lift is extended to the Lorentz group, and in general relativity to the pseudogroup of coordinate transformations. As it stands, this latter extension is well defined only for genuine Riemannian spin manifolds (Euclidean signature) and goes in two steps: (i) from coordinate transformations to local rotations of orthonormal frames, (ii) from these rotations to spin transformations. The second step is double valued and makes use of the spin structure. We would like to know whether this general relativistic lift is unique. By extension, such a lift will be called spin lift.

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In this note we ask the question on infinitesimal level, which is much easier. Indeed for Lie algebras, step (ii) is provided by the very definition of the spin cover and the issue reduces to the uniqueness of the homomorphism from local vector fields to infinitesimal rotations. We prove this uniqueness under the assumption that the homomorphism can be developed as a power series in h for a metric tensor $g = 1 + h$.

The spin lift from Riemannian geometry can be further extended to noncommutative geometry [3–5] where it allows to define the configuration space of the spectral action [2] via the fluctuations of the Dirac operator. This use of the spin lift motivated the question about its uniqueness. In a concluding section, we summarize this motivation together with an open problem that it raises.

2. The pseudogroup of coordinate transformations

Let M be a smooth n-dimensional Riemannian spin manifold with metric g. We are interested in the pseudogroup Γ of (smooth) coordinate transformations between local charts of M defining an atlas. This pseudogroup Γ can be seen as the pseudogroup of local (C^∞-) bidifferential transformations whose Jacobian, expressed in terms of Γ_G-coordinates, belongs to $G = GL(n, \mathbb{R})$. As a (locally flat) continuous pseudogroup, it possesses a Lie pseudoalgebra, which means that the presheaf over M of sections of the sheaf of germs of differentiable vector fields (Γ_G-vector fields over M) has a Lie algebra structure (see for instance [10]).

Here, for simplicity, we prefer to write explicit coordinates in a chart; since we will have to manipulate coordinates with possible singularities, globally defined diffeomorphisms are not sufficient: let us choose an atlas (U_j, α_j) of the manifold M, where the U_j as well as all of their non-empty intersections are contractible open subsets of M. The maps α_j are diffeomorphisms between the U_j and the open subsets $\alpha_j(U_j)$ of $\mathbb{R}^n$. The pseudogroup Γ of coordinate transformations is a collection of the local diffeomorphisms of $\mathbb{R}^n$:

$$\sigma : \mathcal{U} \longrightarrow \mathcal{V},$$

$$\sigma := \alpha_j \circ \alpha_i^{-1} : \quad \alpha_i(U_i \cap U_j) =: \mathcal{U} \longrightarrow \alpha_j(U_i \cap U_j) =: \mathcal{V}.$$

Let $(\tilde{U}_j, \tilde{\alpha}_j)$ be an atlas of the orthonormal frame bundle. We suppose again that the open subsets $\tilde{U}_j$ as well as all of their non-empty intersections are contractible. We also suppose that their projections on M coincide with the U_j.

If our manifold is parallelizable, frames can be defined globally and we can suppose that the $\tilde{\alpha}_j$ are of the form $(\alpha_j, (T\alpha_j)^n)$ where T is the tangent map. Of course, we do not want to restrict ourselves to parallelizable manifolds. Then, $\tilde{\alpha}_j(\tilde{U}_j)$ is a collection of frames defined only over $\alpha_j(U)$. They are orthonormal with respect to the metric induced on this open subset of $\mathbb{R}^n$ from the metric on M by α_j. As before, we define the pseudogroup associated to this atlas whose open sets are the pre-images of non-empty intersections $\tilde{\alpha}_j \circ \tilde{\alpha}_i^{-1} : \tilde{\alpha}_i(\tilde{U}_i \cap \tilde{U}_j) \longrightarrow \tilde{\alpha}_j(\tilde{U}_i \cap \tilde{U}_j)$. Again we simplify notations and write such a diffeomorphism as

$$\tilde{\sigma} : \tilde{\mathcal{U}} \to \tilde{\mathcal{V}}.$$

We denote a generic element in $\tilde{\mathcal{U}}$ by $(x, e(x))$, with $x \in \mathcal{U} \subset \mathbb{R}^n$ and $e(x) \in GL(n, \mathbb{R})$ defining an orthonormal basis $e_a(x)$ by

$$(e_1(x), \ldots, e_n(x)) := e^{-1}(x) \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right),$$

where orthonormality is with respect to the pull back of g on M to $\mathcal{U}$ by α_i^{-1}.

The metric tensor of this induced metric with respect to the coordinates x^μ is written

$$g_{\mu\nu}(x) = g\left((T\alpha_i)^{-1}\frac{\partial}{\partial x^\mu},(T\alpha_i)^{-1}\frac{\partial}{\partial x^\nu}\right)\bigg|_{\alpha_i(x)}.$$

Following physicists' dangerous tradition, we use the same letter e to denote orthonormal frames and their coefficients with respect to the homomorphisms of $\mathbb{R}^n$ and we use the same letter g to denote the metric as a two tensor field and its local representation as a matrix. Then, orthonormality of e, $g(e_i(x), e_j(x)) = \delta_{ij}$, is equivalent to the matrix equation $e^{-1T} g\, e^{-1} = 1_n$ and in the following, unless specified, g will always be seen as a matrix.

We would like to write the transformation $\tilde{\sigma}$ in these notations. If M is not parallelizable, we must include gauge transformations, i.e. the coordinate transformations are now modified by the introduction of local rotations Λ:

$$\tilde{\sigma} := (\sigma, (T_x\sigma)^n \circ \Lambda(x)), \quad \Lambda : \mathcal{U} \longrightarrow SO(n).$$

Let us alleviate notations by dropping the tangent maps $(T_x\sigma)^n$ from the right hand side. Then the composition law reads:

$$(\tau, \Omega) \circ (\sigma, \Lambda) = (\tau \circ \sigma, (\Omega \circ \sigma)\Lambda), \tag{2.1}$$

whenever target and source match. In terms of the frame bundle, this is the cocycle condition. If targets and sources coincide, we just have the multiplication law in the semi-direct product, $\mathrm{Diff}(\mathcal{U}) \ltimes {}^{\mathcal{U}}SO(n)$, where by ${}^{\mathcal{U}}K$ we denote the group of smooth functions from $\mathcal{U}$ to the group K.

Definition 1. A *pseudogroup homomorphism* between Γ and the pseudogroup of local frame rotations is a collection of morphisms $L : \sigma \in \mathrm{Diff}(\mathcal{U}, \mathcal{V}) \mapsto (\sigma, \Lambda(\sigma, g)) \in \mathrm{Diff}(\mathcal{U}, \mathcal{V}) \ltimes {}^{\mathcal{U}}SO(n)$, respecting compositions whenever possible.

The projection $p : (\sigma, \Lambda) \mapsto \sigma$ is a homomorphism from the pseudogroup of gauge transformations to the pseudogroup of coordinate transformations.

A lift for p is a map L constructed from the metric only and such that $L \circ p = \mathrm{Id}$.

The dependence on the metric is indicated in $\Lambda(\sigma, g)$ through the metric tensor g on the source $\mathcal{U}$ of σ.

We denote by $\mathcal{J}_\sigma(x)^\nu_{\ \mu} := \partial\sigma^\nu(x)/\partial x^\mu$ the Jacobian of the coordinate transformation σ and recall the composition law $\mathcal{J}_{\tau\circ\sigma} = (\mathcal{J}_\tau \circ \sigma)\mathcal{J}_\sigma$. Let us write

$$\sigma \cdot g := (\mathcal{J}_\sigma^{-1T} g\, \mathcal{J}_\sigma^{-1}) \circ \sigma^{-1}$$

(sometimes denoted by $(\sigma^{-1})^* g$) for the push forward from $\mathcal{U}$ to $\mathcal{V}$ of the metric tensor by σ (pull back by σ^{-1}). Then the composition law takes the form

$$(\tau \circ \sigma) \cdot g = \tau \cdot (\sigma \cdot g)$$

and we have the

Lemma 2. *A lift L is a pseudogroup homomorphism if and only if*

$$\Lambda(1, g) = 1_n, \quad \Lambda(\tau \circ \sigma, g) = (\Lambda(\tau, \sigma \cdot g) \circ \sigma)\Lambda(\sigma, g). \tag{2.2}$$

An immediate question is of course to construct a solution to (2.2).

Here is a construction of such a lift based on a canonical way to orthonormalize a basis b_k: Denote by $g_{k\ell} := g(b_k, b_\ell)$ the matrix of scalar products and remark that g is a positive matrix and that $\sqrt{g}^{-1} = \sqrt{g^{-1}}$ since g^{-1} and g commute. Now, define the orthonormal basis

$$(e_1(x), \ldots, e_n(x)) := \sqrt{g}^{-1}(x)(b_1(x), \ldots, b_n(x)).$$

Naturally, the matrix $\sqrt{g}$ has no immediate geometric link with the original metric corresponding to the matrix g.

Applying this procedure to the holonomic basis $\partial/\partial x^\mu$ over $\mathcal{U}$, we get the orthonormal frame e with coefficient matrix $e^{-1} := \sqrt{g}^{-1}$. Following Raymond Stora we call this *the symmetric gauge* because the coefficient matrix e^{-1} is symmetric [1,9].

In the same manner we get an orthonormal frame f over the target $\mathcal{V}$ and the gauge transformations $\Lambda(\sigma, g)$ is the unique rotation that compares the orthonormal basis $\{e_a\}_a$ and the pullback one $\{T\sigma^{-1} f_b\}_b$, namely $e_a =: {\Lambda^{-1}}^b{}_a T\sigma^{-1} f_b$.

Explicitly this gauge transformation is

$$\Lambda(\sigma, g)|_x = [\sqrt{\mathcal{J}_\sigma^{-1T}\, g\, \mathcal{J}_\sigma^{-1}}\, \mathcal{J}_\sigma \sqrt{g^{-1}}]_x. \tag{2.3}$$

This is the first step, from coordinate transformation to local rotations. The second step to spin transformations is well known. The composition of the two steps yields the desired spin lift $\mathbb{L}(\sigma, g) = (\sigma, S(\Lambda(\sigma, g)))$ where S is the usual double valued homomorphism

$$\begin{cases} S : SO(n) \longrightarrow Spin(n), \\ \Lambda = \exp\omega \longmapsto \exp(\frac{1}{8}\omega_{ab}[\gamma^a, \gamma^b]) \end{cases} \tag{2.4}$$

with $\omega = -\omega^T \in so(4)$. More explicitly, we get for any spinor ψ and $x \in \mathcal{U} = \text{Source}(\sigma)$

$$(\sigma, \Lambda(\sigma, g))\psi(\sigma(x)) = S(\Lambda(\sigma, g))(x)\psi(x).$$

The spin structure on M allows to glue $\mathbb{L}$ together consistently.

If M is flat $\mathbb{R}^3$ in Cartesian coordinates and σ is a (rigid) rotation R, then $\Lambda = R$ and $\mathbb{L}$ is the spin lift of quantum mechanics.

A natural question arises: are there other lifts $\mathbb{L}$, or an easier question: are there gauge transformations $\Lambda(\sigma, g)$ other than (2.3) satisfying the multiplication law (2.2)?

We address the second question on infinitesimal level.

3. The Lie pseudoalgebra of vector fields

3.1. Linearizing the composition law

Consider the corresponding homomorphism of Lie pseudoalgebras:

$$\sigma(x) = x + \xi(x), \quad \tau(x) = x + \eta(x), \quad \Lambda(\sigma, g) = 1_n + \lambda(\xi, g) + \tfrac{1}{2}\lambda(\xi, g)^2 + \cdots \tag{3.1}$$

where ξ and η are vector fields on $\mathcal{U}$ and the infinitesimal rotations $\lambda \in so(n)$ are linear in the vector field. For infinitesimal coordinate transformations, the composition law (2.2) reads

$$\begin{aligned}\lambda([\xi, \eta], g) = {}& [\lambda(\eta, g), \lambda(\xi, g)] - L_\eta\lambda(\xi, g) + L_\xi\lambda(\eta, g) \\ & -\frac{\delta}{\delta g}\lambda(\xi, g)\delta_\eta g + \frac{\delta}{\delta g}\lambda(\eta, g)\delta_\xi g\end{aligned} \tag{3.2}$$

where we write the functional derivative with respect to the metric tensor as

$$\frac{\delta}{\delta g}\lambda(\xi, g)\delta g := \lambda(\xi, g + \delta g) - \lambda(\xi, g) + O(\delta g^2). \tag{3.3}$$

The variation of the metric tensor under an infinitesimal coordinate transformation is denoted by

$$\delta_\xi g := -\frac{\partial \xi}{\partial x}^T g - g\frac{\partial \xi}{\partial x} - L_\xi g,$$

the metric tensor g being considered as matrix valued 0-form.

3.2. Linearizing the natural solution

To linearize Eq. (2.3), the following lemma will be used repetitively.

Lemma 3. *Let X be a positive $n \times n$ matrix. Then, the solution of the constraint*

$$\sqrt{g+X} = \sqrt{g} + \tfrac{1}{2}\sqrt{g}^{-1} I_g(X) + O(X^2)$$

is

$$I_g(X) := \int_{-\infty}^{\infty} \sqrt{g}^{it+(1/2)} X \sqrt{g}^{-it-(1/2)} \frac{\mathrm{d}t}{\cosh(\pi t)}. \tag{3.4}$$

Proof. We first remark that a solution of the equation $AY + YA = B$ where A, B are positive matrices with A invertible is

$$Y = \frac{1}{2}\int_{-\infty}^{\infty} A^{it-(1/2)} B A^{-it-(1/2)} \frac{\mathrm{d}t}{\cosh(\pi t)}.$$

as can be checked directly using complex integration (see for instance [7]). This solution is unique: if $Z = Z^*$ (as we may assume) is the difference of two solutions, $AZ + ZA = 0$, thus if P_+ is the projection on the positive part Z_+ of Z, $P_+AZ_+ + Z_+AP_+ = 0$, $A^{1/2}Z_+A^{1/2} = 0$ since the spectra of P_+AZ_+ and $A^{1/2}Z_+A^{1/2}$ are equal (up to a possible zero) and $Z_+ = 0$; similarly $Z_- = 0$.

Using analytical properties of the square root, we can write $\sqrt{g+X} = \sqrt{g} + B + O(X^2)$ with $\|B\| \le c\|X\|$. Thus $g + X = (\sqrt{g}+B)(\sqrt{g}+B) + O(X^2) = g + \sqrt{g}B + B\sqrt{g} + B^2 + O(X^2)$ and $X = \sqrt{g}B + B\sqrt{g}$, so $B = \frac{1}{2}\sqrt{g}^{-1} I_g(X)$ according to the previous remark. □

A few useful formulas are

$$I_{1_n}(X) = X, \quad I_{g^{-1}}(X) = \sqrt{g}^{-1} I_g(X)\sqrt{g}, \quad \int_{-\infty}^{\infty} t^{2\ell} \frac{\mathrm{d}t}{\cosh(\pi t)} = \frac{E_\ell}{2^{2\ell}},$$

with the Euler numbers $E_1 = 1$, $E_2 = 5$, $E_3 = 61, \ldots$

Now, the linearized Eq. (2.3) can be written as

$$\lambda(\xi, g) = \tfrac{1}{2}\sqrt{g}^{-1} I_g\left(-\frac{\partial \xi}{\partial x}^T g - g\frac{\partial \xi}{\partial x}\right)\sqrt{g}^{-1} + \sqrt{g}\frac{\partial \xi}{\partial x}\sqrt{g}^{-1}.$$

However, strangely enough, both infinitesimal versions, that of the composition law and that of the natural solution, are more involved than their finite versions. In order to continue nevertheless, we will resort to perturbation theory around the flat metric.

To this end, using the abbreviation

$$\delta_\xi^- g := -\frac{\partial \xi}{\partial x}^T g - g\frac{\partial \xi}{\partial x},$$

the functional derivative (3.3) of the natural solution λ with respect to the metric tensor is

$$\begin{aligned}\frac{\delta}{\delta g}\lambda(\xi, g)\delta g = &-\frac{1}{2}\sqrt{g}\frac{\partial\xi}{\partial x}g^{-1}I_g(\delta g)\sqrt{g}^{-1} + \frac{1}{2}\sqrt{g}^{-1}I_g(\delta g)\frac{\partial\xi}{\partial x}\sqrt{g}^{-1}\\ &+\frac{1}{2}\sqrt{g}^{-1}I_g(\delta_\xi^- \delta g)\sqrt{g}^{-1} - \frac{1}{4}\sqrt{g}^{-1}I_g(\delta_\xi^- g)g^{-1}I_g(\delta g)\sqrt{g}^{-1}\\ &-\frac{1}{8}\sqrt{g}^{-1}I_g(\sqrt{g}^{-1}I_g(\delta g)\sqrt{g}^{-1}I_g(\delta_\xi^- g))\sqrt{g}^{-1}\\ &-\frac{1}{8}\sqrt{g}^{-1}I_g(\sqrt{g}^{-1}I_g(\delta_\xi^- g)\sqrt{g}^{-1}I_g(\delta g))\sqrt{g}^{-1}.\end{aligned}$$

Now we can compute its Taylor series around the unit metric tensor

$$\begin{aligned}\lambda(\xi, 1_n + h) = &\frac{1}{2}\left(\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial x}^T\right) + \frac{1}{8}h\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right) - \frac{1}{8}\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right)h\\ &-\frac{1}{16}h^2\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right) + \frac{1}{16}\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right)h^2 + O(h^3). \qquad (3.5)\end{aligned}$$

3.3. Uniqueness of the Taylor series

Theorem 4. *Any Taylor series satisfying the infinitesimal composition law (3.2) is either identically zero or coincides with the Taylor series of the natural solution (3.5).*

Proof. Let us view the commutator (3.2) as a first order functional differential equation. We may hope to have a unique solution if we have an initial condition at $g = 1_n$. This initial condition $\lambda(\xi, 1_n)$ is an antisymmetric $n \times n$ matrix, linear in ξ and covariant under rigid rotations of the Cartesian coordinates x:

We suppose that the solution $\lambda(\xi, g)$ can be written as a Taylor series in the components of ξ and g and of their derivatives. Then the initial condition, $g = 1_n$, does not contain derivatives of the metric tensor and it must be of the form

$$\lambda(\xi, 1_n) = a\left(\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial x}^T\right) + a_1\left(\frac{\partial\Delta\xi}{\partial x} - \frac{\partial\Delta\xi}{\partial x}^T\right) + a_2\left(\frac{\partial\Delta^2\xi}{\partial x} - \frac{\partial\Delta^2\xi}{\partial x}^T\right) + \cdots$$

where Δ is the (flat) Laplacian. The first derivative $\frac{\delta}{\delta g}\lambda(\xi, 1_n)\delta g$ is an antisymmetric matrix, linear in ξ and δg:

$$\begin{aligned}\frac{\delta}{\delta g}\lambda(\xi, 1_n)\delta g = &\left(b\frac{\partial\xi}{\partial x} + c\frac{\partial\xi}{\partial x}^T\right)\delta g + \left(b_{11}\frac{\partial\Delta\xi}{\partial x} + c_{11}\frac{\partial\Delta\xi}{\partial x}^T\right)\delta g\\ &+\left(b_{12}\frac{\partial\xi}{\partial x} + c_{12}\frac{\partial\xi}{\partial x}^T\right)\Delta\delta g + \left(b_{13}\frac{\partial\xi_\alpha}{\partial x} + c_{13}\frac{\partial\xi_\alpha}{\partial x}^T\right)\delta g_\alpha\\ &-\text{transposed} + \text{higher derivatives}.\end{aligned}$$

Plugging these two series into Eq. (3.2) with $g = 1_n$ leaves us with two solutions for the coefficients $a, b, c, \ldots$ There is the trivial solution where all coefficients vanish. The other solution is

$$a = \tfrac{1}{2}, \quad b = c = -\tfrac{1}{8},$$

all higher coefficients vanish. We now have our initial condition

$$\lambda(\xi, 1_n) = \frac{1}{2}\left(\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial x}^T\right)$$

and Eq. (3.2) determines its first derivative,

$$\frac{\delta}{\delta g}\lambda(\xi, 1_n)\delta g = \frac{1}{8}\delta g\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right) - \frac{1}{8}\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right)\delta g.$$

In other words we know

$$\lambda(\xi, 1_n + h) = \frac{1}{2}\left(\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial x}^T\right) + \frac{1}{8}h\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right) - \frac{1}{8}\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right)h + O(h^2).$$

To get the terms of order h^2, we use again Eq. (3.2) with $g = 1_n + h = 1_n + \delta g_2$ and compute the second functional derivative,

$$\frac{\delta^2}{\delta g_1 \delta g_2}\lambda(\xi, 1_n)\,\delta g_1 \delta g_2 := \frac{\delta}{\delta g_1}\lambda(\xi, 1_n + \delta g_2)\,\delta g_1 - \frac{\delta}{\delta g_1}\lambda(\xi, 1_n)\,\delta g_1 + O(\delta g_2^2).$$

Therefore,

$$\begin{aligned}\lambda(\xi, 1_n + h) = {}& \frac{1}{2}\left(\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial x}^T\right) + \frac{1}{8}h\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right) - \frac{1}{8}\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right)h \\ & - \frac{1}{16}h^2\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right) + \frac{1}{16}\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}^T\right)h^2 + O(h^3),\end{aligned}$$

reproducing the beginning of the Taylor series (3.5). We also see that Eq. (3.2) fixes the terms of order h^{N+1} in $\lambda(\xi, 1_n + h)$ from the terms of order h^N, which concludes the proof. □

Remark 5.

- As pointed out in Section 2, the trivial solution only makes sense on parallelizable manifolds and, although used in the physics literature, we must discard it. So, in this restrictive sense, the Lie pseudoalgebra homomorphism is unique.
- Note that all terms of the series are local in the sense that they contain no derivatives of g or equivalently of h with respect to x.

Of course a proof for finite diffeomorphisms, and without resorting to power series remains desirable.

4. Lift and noncommutative geometry

In this concluding section we would like to explain our motivation concerning the uniqueness of the spin lift and an interesting, open problem it leads to.

We note that the spin lift from general relativity can be further extended to noncommutative geometry [3–5] where it then allows to define the configuration space of the spectral action [2] via the fluctuations of the Dirac operator. For inner automorphisms, these fluctuations have a natural motivation from Morita equivalence, but they continue to make sense for outer automorphisms like diffeomorphisms of a Riemannian spin manifold. In this commutative case, the spectral action reproduces general relativity with a positive cosmological constant plus a curvature square term. For almost commutative geometries, the spectral action produces, in addition, the complete Yang–Mills–Higgs action e.g. the standard model of electromagnetic, weak and strong forces [2].

The commutative case relies on three steps:

- Connes' reconstruction theorem [5] that motivates the definition of spectral triples.
- A result on the uniqueness of the extension of the spin lift to all diffeomorphisms.
- A result that puts Einstein's equivalence principle on a mathematical footing by the use of Connes' fluctuating metric.

The guiding example to noncommutative geometry is any n-dimensional, compact Riemannian spin manifold M. It defines a real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with algebra $\mathcal{A} = \mathcal{C}^\infty(M)$ faithfully represented on the Hilbert space $\mathcal{H}$ of square integrable Dirac spinors and the selfadjoint Dirac operator $\mathcal{D} = \partial\!\!\!/$ possibly with torsion. The real structure is defined by the anti-unitary operator J that physicists call charge conjugation when M is interpreted as space-time. For even dimensional manifolds M there is another operator χ defining a $\mathbb{Z}_2$-grading of the Hilbert space. In physics it is called chirality, $\chi = \gamma_5$.

Some of the properties of the four or five items $\mathcal{A}, \mathcal{H}, \mathcal{D}, J, (\chi)$ are promoted to axioms of the (even), real spectral triple. The commutativity of the algebra $\mathcal{A} = \mathcal{C}^\infty(M)$ is not promoted and the triple is called commutative if its algebra is. The selection of the promoted properties is motivated by Connes' reconstruction theorem [5]. It states that every commutative, (even), real spectral triple comes from an (even dimensional) Riemannian spin manifold. An explicit proof of a weaker form of the reconstruction theorem can be found in the Costa Rica book [6, Theorem 11.2].

If M is interpreted as phase space, then $\mathcal{A}$ is the algebra of classical observables. Not promoting its commutativity opens the door to Heisenberg's uncertainty relation.

In general relativity, Einstein generalizes rotations to general coordinate transformations. Let us momentarily ignore coordinate singularities and confuse coordinate transformation and diffeomorphism. In the language of spectral triples, diffeomorphisms are algebra automorphisms, $\mathrm{Aut}(\mathcal{C}^\infty(M)) = \mathrm{Diff}(M)$. Our aim is therefore to lift any algebra automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$ to unitaries on the Hilbert space $\mathcal{H} : \sigma \mapsto \mathbb{L}(\sigma)$. In addition to being unitary we want the lifted automorphism to commute with the real structure and in the even case with the chirality, and to satisfy the covariance property

$$\mathbb{L}(\sigma)\rho(a)\mathbb{L}(\sigma)^{-1} = \rho(\sigma(a)), \quad \text{for all}\, a \in \mathcal{A}, \tag{4.1}$$

where we denote by ρ the faithful representation of $\mathcal{A}$ on $\mathcal{H}$. This property allows to define the projection of the lift back to the automorphism group

$$p(\mathbb{L}(\sigma))(a) = \rho^{-1}(\mathbb{L}(\sigma)\rho(a)\mathbb{L}(\sigma)^{-1}). \tag{4.2}$$

Of course we also want $\mathbb{L}$ to be a group homomorphism, possibly multi-valued.

In the commutative case, if U is a coordinate neighborhood in M we have for diffeomorphisms close to the identity

$$\mathbb{L}(\mathrm{Diff}(U)) = \mathrm{Diff}(U) \ltimes {}^{U}Spin(n). \tag{4.3}$$

In coordinates a possible lift is $\mathbb{L} = (1, S \circ \Lambda)$ where S is the usual double valued homomorphism (2.4).

4.1. Equivalence principle

Einstein uses the equivalence principle to guess the configuration space of general relativity: His starting point is the flat metric in inertial coordinates. With a general coordinate transformation he arrives at the metric tensor relevant for a uniformly accelerated observer. For this observer, a free, massive point-particle is subject to a constant pseudo-force, which looks like gravity on earth. Therefore, he proposes nontrivial metric tensors to encode gravity. If we want to formulate Einstein's idea in mathematical terms, we face the problem that a flat metric will still be flat after a general coordinate transformation and to express the equivalence between the free particle as seen by the accelerated observer and the particle falling freely in a static gravitational field as seen by an observer at rest with respect to the gravitational field we need more than one coordinate transformation.

This situation is somehow reminiscent of special relativity and electromagnetism. The magnetic field of a constant rectilinear current is a pseudo-force, which can be transformed to zero with the boost that puts the observer at rest with respect to the charges. A general magnetic field could be viewed as generated by a superposition of many small rectilinear currents and one is tempted to use many 'local' boosts to transform it to zero.

In a sense this is what Alain Connes does when he *fluctuates* the metric. In his reconstruction theorem, the metric is reconstructed from and therefore encoded in the Dirac operator. Now Connes replaces the point-like matter in Einstein's reasoning by a Dirac particle. Then the Dirac operator $\mathcal{D}$ plays two roles simultaneously, (i) it defines the metric and therefore the initial, say zero, gravitational field, (ii) it defines the dynamics of matter. Using the spin lift $\mathbb{L}(\sigma)$ we can act with a general coordinate transformation on the Dirac particle $\psi \in \mathcal{H}$. In a coordinate neighborhood, this action reads

$$(\mathbb{L}(\sigma)\psi)(x) = S(\Lambda(\sigma, g))|_{\sigma^{-1}(x)}\psi(\sigma^{-1}(x)), \tag{4.4}$$

and amounts to replacing the initial flat Dirac operator $\mathcal{D}$ by the still flat operator $\mathbb{L}(\sigma)\mathcal{D}\mathbb{L}(\sigma)^{-1}$. So far we have not gained anything with respect to Einstein's point of view. However, unlike metrics, Dirac operators can be linearly combined and Connes defines the fluctuations of the initial metric or Dirac operator as the finite linear combinations

$$\sum r_j \mathbb{L}(\sigma_j)\mathcal{D}\mathbb{L}(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \mathrm{Aut}(\mathcal{A}). \tag{4.5}$$

We arrive naturally at the following

Problem 6. Given two Dirac operators with arbitrary curvature and torsion defined on the same manifold, can one be written as a fluctuation of the other?

Of course, the configuration space of general relativity, the space of all gravitational fields, should be identified as the *affine* space of all fluctuations of the initial Dirac operator, from which however we delete those fluctuations that do not define a spectral triple, e.g. the fluctuation that is identically zero.

For almost commutative spectral triples this affine space in addition contains the Yang–Mills connections and the Higgs scalar [4,5].

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